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Critical properties of Dyson's hierarchical model II. Essential singularities of the borderline Ising case

D Kim and C J Thompson

Mathematics Department, University of Melbourne, Parkville, Victoria 3052, Australia

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Abstract. The borderline case of Dyson's hierarchical model where a first-order transition is known rigorously to occur is investigated numerically. The susceptibility is found to behave as $\chi \sim \exp[a(\beta_c-\beta)^{-1}]$ as $\beta \to \beta_c-$ and infinite for $\beta > \beta_c$. The spontaneous magnetisation and correlation functions are also investigated. In terms of renormalisation group theory, the essentially singular behaviour results from the relevant operator becoming marginal in the borderline case.

1. Introduction and summary

The one-dimensional Ising model with long-range interaction whose potential falls off like $r^{-(1+\sigma)}$ where r is the distance between two interacting spins has a phase transition for $0 < \sigma < 1$ and no phase transition for $\sigma > 1$ (Dyson 1969). At the borderline $\sigma = 1$, there is no rigorous proof concerning the existence of a phase transition. Thouless (1969), however, has argued that the phase transition, if it exists, should be of first order; that is, a jump discontinuity in the spontaneous magnetisation at a finite critical temperature T_c. This suggestion was brought out in an approximate renormalisation group calculation by Anderson and Yuval (1971) who also concluded that the free energy has an essential singularity at T_c. A similar approach by Kosterlitz (1976) further indicated that the zero-field susceptibility χ has an essential singularity of the form

$$\chi \sim \exp[a(T-T_c)^{-1/2}] \qquad \text{as } T \to T_c+. \tag{1.1}$$

Our aim here is to investigate the borderline hierarchical model (HM), which is known rigorously to undergo a first-order phase transition (Dyson 1971). This model, for a system of 2^N Ising spins, is defined in general by the Hamiltonian

$$\mathcal{H}_N = -\sum_{p=1}^{N} 2^{-2p} b_p \sum_{r=1}^{2N-p} S_{p,r}^2 - HS_{N,1} \tag{1.2}$$

where

$$S_{p,r} = S_{p-1,2r-1} + S_{p-1,2r} = \sum_i S_{0,i} \tag{1.3}$$

for $(r-1)2^p + 1 \leq i \leq r2^p$ and $S_{0,i} = \pm 1$.

When $b_p = 2^{(1-\sigma)p}$ the HM immitates the power law $r^{-(1+\sigma)}$ Ising chain. The borderline case referred to above and with which we will be concerned in this paper is for $b_p = \ln p$.

In the following sections we use recursion relations for the partition function (Kim and Thompson 1977) and correlation functions, to calculate the high-temperature susceptibility, the spontaneous magnetisation, and correlations for various values of $\beta = (k_B T)^{-1}$.

Numerically we find that the susceptibility has an essential singularity of the form

$$\chi \sim \exp[a(\beta_c - \beta)^{-1}] \qquad \text{as } \beta \to \beta_c - \tag{1.4}$$

and is infinite for all $\beta > \beta_c \simeq 1{\cdot}71155 \pm 0{\cdot}0001$. This is to be compared with (1.1). We also obtain spontaneous magnetisation curves which are unfortunately not sufficiently accurate for β near β_c to do better than the estimate $m_{0c} = 0{\cdot}5 \pm 0{\cdot}15$ for the magnitude of the jump discontinuity in the spontaneous magnetisation at β_c. This is to be compared with Dyson's (1971) rigorous lower bound of $(1 + \frac{4}{3}\beta_c)^{-1} \simeq 0{\cdot}30$ and the Anderson and Yuval (1971) computed value of 0·89 for the borderline Ising chain. The correlation function $C(p)$ between two spins coupled at the pth level of the hierarchy is found to decay as

$$C(p) \sim \begin{cases} \chi^2 2^{-2p} \ln p & \text{for } \beta < \beta_c \quad \text{as } p \to \infty \\ (\ln p)^{-1} & \text{for } \beta = \beta_c \quad \text{as } p \to \infty. \end{cases} \tag{1.5}$$

In order to understand the occurrence of essential singularities within the framework of renormalisation group theory, an argument is presented in the final section showing that near a fixed point of the renormalisation group transformation (RGT) for a borderline situation, the first relevant parameter of the RGT, corresponding to maximum eigenvalue unity of the linearised transformation, deviates *linearly* rather than *exponentially* from its fixed point value as the RGT evolves. We show that this linear behaviour results in essential singularities of the form (1.4). Special features of the borderline HM combined with the argument presented in this section are consistent with a first-order transition.

2. Susceptibility and order parameter

To determine the susceptibility χ for the HM (1.2) we consider the moment generating function for the mean magnetisation defined by

$$T_N(h) = \langle \exp(m_N h) \rangle_N \tag{2.1}$$

where

$$m_N = 2^{-N} S_{N,1} = 2^{-N} \sum_{i=1}^{2N} S_{0,i} \tag{2.2}$$

is the magnetisation per spin and the average in (2.1) is taken with respect to the zero-field ($H = 0$) Hamiltonian (1.2). Alternatively, in terms of the partition function

$$Q_N(\beta, h) = \sum_{\{S_{0,i} = \pm 1\}} \exp(-\beta \mathcal{H}_N) \tag{2.3}$$

with $h = \beta H$, we have

$$T_N(h) = Q_N(\beta, 2^{-N} h) / Q_N(\beta, 0). \tag{2.4}$$

Using the recursion relation for the partition function (Kim and Thompson 1977)

$$Q_N(\beta, h) = \pi^{-1/2} \int_{-\infty}^{\infty} e^{-x^2}[Q_{N-1}(\beta, h + 2(\beta b_N)^{1/2} 2^{-N} x)]^2 \, dx \qquad (2.5)$$

and approximating $T_N(h)$ by a polynomial in h of order $2M$, i.e.

$$T_N(h) \approx 1 + \sum_{k=1}^{M} A_{k,N}(\beta) h^{2k}/(2k)! \qquad (2.6)$$

the coefficients $A_{k,N}(\beta)$ can be readily generated on a computer using equations (2.9) and (2.5) for successive values of N. By definition, when $\beta < \beta_c$,

$$\chi(\beta) = \lim_{N\to\infty} \chi_N(\beta) \qquad (2.7)$$

where

$$\chi_N(\beta) = 2^N A_{1,N}(\beta) \qquad (2.8)$$

so that the suceptibility is obtained directly by extrapolating the $\{A_{1,N}(\beta)\}$ sequence. As a practical procedure, convergence can be accelerated by noting that for sufficiently large N

$$T_N(h) \approx \exp(A_{1,N}(\beta) h^2/2). \qquad (2.9)$$

Using (2.9), (2.4) and (2.5) we then have

$$\chi_{N+1}^{-1} \approx \chi_N^{-1} - \beta b_{N+1} 2^{-N}$$

from which it follows that

$$\chi^{-1} \approx \chi_N^{-1} - \beta 2^{-N} \sum_{k=0}^{\infty} 2^{-k} b_{N+1+k} \approx \chi_N^{-1} - 2\beta 2^{-N} \ln(N+1), \qquad (2.10)$$

where in the last step we have used the borderline value $b_p = \ln p$.

The values of $\chi(\beta)$ obtained using the above prescription with $M = 40$ are given in table 1 for a range of β near criticality. In figure 1 we have plotted the inverse of $\log_2 \chi$ against β from which we conclude that

$$(\log_2 \chi)^{-1} \approx A(\beta_c - \beta) \qquad \text{as } \beta \to \beta_c - \qquad (2.11)$$

Table 1. Logarithm of the high-temperature susceptibility data near the critical temperature.

β	$\log_2 \chi$	β	$\log_2 \chi$
1·60	9·269	1·700	44·078
1·61	9·810	1·702	52·150
1·62	10·440	1·704	64·808
1·63	11·192	1·706	87·343
1·64	12·109	1·708	137·269
1·65	13·268	1·710	315·181
1·66	14·802		
1·67	16·968		
1·68	20·359		
1·69	26·700		

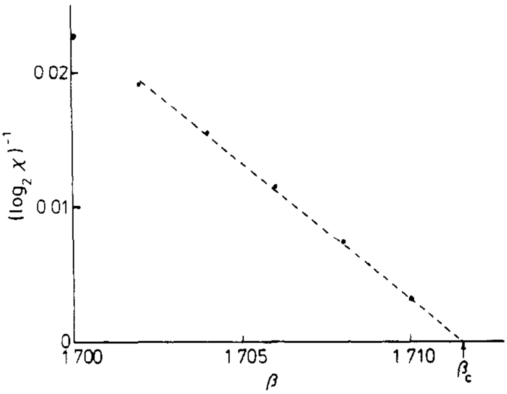

Figure 1. $(\log_2 \chi)^{-1}$ against β near β_c from table 1. The broken line is for visual aid.

with

$$\beta_c = 1{\cdot}71155 \pm 0{\cdot}0001 \qquad \text{and} \qquad A \approx 2{\cdot}0 \pm 0{\cdot}2. \tag{2.12}$$

In other words the high-temperature susceptibility has an essential singularity of the form

$$\chi(\beta) \sim \exp[a(\beta_c - \beta)^{-1}] \qquad \text{as } \beta \to \beta_c - \tag{2.13}$$

with $a \approx 0{\cdot}35$.

When $\beta > \beta_c$ we expect the $A_{k,N}$ to approach finite limits as $N \to \infty$. In fact, following Griffiths (1966) we can define a sequence of order parameters m_{2k} though

$$m_{2k} = \lim_{N\to\infty} (\langle m_N^{2k} \rangle_N)^{1/2k} = \lim_{N\to\infty} (A_{k,N})^{1/2k}. \tag{2.14}$$

With small numerical uncertainty we find that for all k the m_{2k} are the same as the order parameter m_0 defined by

$$m_0^2 = \lim_{p\to\infty} \lim_{N\to\infty} \langle S_{0,1} S_{0,2p} \rangle_N. \tag{2.15}$$

For example at $\beta = 1{\cdot}8$ we find that $m_{2k} = m_0 = 0{\cdot}8122 \pm 0{\cdot}0001$ for $k = 1, 2, \ldots$. (The procedure used to obtain m_0 is discussed in the following section.)

In figure 2 we have shown the 'spontaneous magnetisation curve' m_{2k} for a range of β near β_c obtained by extrapolating the $A_{k,N}$ sequences for several k.

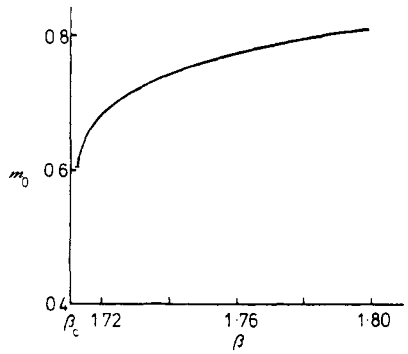

Figure 2. Spontaneous magnetisation m_0 against β.

Unfortunately, the $A_{k,N}$ sequences converge very slowly for β close to β_c and we could only roughly estimate that the magnitude of the jump in the spontaneous magnetisation at β_c is approximately $m_{0c} \approx 0{\cdot}5 \pm 0{\cdot}15$. There is little doubt that the critical value is approached with infinite slope as $\beta \to \beta_c+$ but with the numerical uncertainty we are unable to reliably estimate the nature of the singularity.

3. Correlations

Following Dyson (1969), we define the correlation function $C_N(p)$ between two spins coupled at the pth level of the hierarchy by

$$C_N(p) = \langle S_{0,1} S_{0,2^p} \rangle_N. \tag{3.1}$$

To obtain exact recursion relations we introduce

$$\tilde{C}_{N,p}(y) = \langle S_{0,1} S_{0,2^p} \exp(m_N y) \rangle_N \tag{3.2}$$

where $m_N = 2^{-N} S_{N,1}$ as before. Following the same line of argument used to derive the recursion relation (2.5) for the partition function (Kim and Thompson 1977) we obtain, for $N > p$,

$$\tilde{C}_{N,p}(2y) = [Q_{N-1}(\beta, 0)]^2 [Q_N(\beta, 0)(\pi b_N)^{1/2}]^{-1} \times \int_{-\infty}^{\infty} \exp[-(x-y)^2/\beta b_N] \tilde{C}_{N-1,p}(x) T_{N-1}(x)\,\mathrm{d}x \tag{3.3}$$

where $b_N = \ln N$ and $T_N(x)$ is defined by (2.1) or (2.4). It follows that once $\tilde{C}_{p,p}(y)$ is known, $\tilde{C}_{N,p}(y)$ for $N > p$ can be readily generated by successive application of (3.3), using a similar procedure to that described in the previous section for $T_N(y)$. Also since $C_N(p) = \tilde{C}_{N,p}(0)$, the limiting correlation function

$$C(p) = \lim_{N \to \infty} C_N(p) \tag{3.4}$$

can then be obtained by extrapolating the $C_N(p)$ sequences.

To determine the starting point $\tilde{C}_{p,p}(y)$ for (3.3) we observe that from (2.1)

$$\begin{aligned}\frac{\mathrm{d}^2 T_p(y)}{\mathrm{d}y^2} &= \langle m_p^2 \exp(m_p y) \rangle_p \\ &= \langle \tfrac{1}{2} m_{p-1}^2 \exp(m_p y) \rangle_p + 2^{-2p+1} \langle S_{p-1,1} S_{p-1,2} \exp(m_p y) \rangle_p \\ &= \langle \tfrac{1}{2} m_{p-1}^2 \exp(m_p y) \rangle_p + 2^{-2p+1} \sum_{i=1}^{2^{p-1}} \sum_{j=2^{p-1}+1}^{2^p} \langle S_{0,i} S_{0,j} \exp(m_p y) \rangle_p.\end{aligned} \tag{3.5}$$

Due to the hierarchical structure of the model each term in the sum appearing in (3.5) is equal to $\tilde{C}_{p,p}(y)$ so that on re-arranging we have

$$\tilde{C}_{p,p}(y) = 2 \frac{\mathrm{d}^2 T_p(y)}{\mathrm{d}y^2} - \langle m_{p-1}^2 \exp(m_p y) \rangle_p. \tag{3.6}$$

The second term in (3.6) can be expressed in term of T_{p-1} as

$$\langle m_{p-1}^2 \exp(m_p y)\rangle_p = [Q_{p-1}(\beta, 0)]^2 [Q_p(\beta, 0)(\pi\beta b_p)^{1/2}]^{-1} \times \int_{-\infty}^{\infty} \exp[-(x-y/2)^2/\beta b_p] T_{p-1}(x) \frac{d^2 T_{p-1}(x)}{dx^2} dx \tag{3.7}$$

where $b_p = \ln p$. Therefore, $\tilde{C}_{p,p}(y)$ is determined from $T_{p-1}(y)$ and $T_p(y)$ which in turn are determined recursively from (2.4) and (2.5).

Numerically, the $C_N(p)$ sequences converge rapidly for fixed p and increasing N. In practice $C_{p+30}(p)$ were sufficient to obtain six digit accuracy for $C(p)$ for values of p up to 400.

For high temperatures ($\beta < \beta_c$) the $C(p)$ data give

$$C(p) \sim 2^{-2(p-L_0)} \ln p \qquad \text{as } p \to \infty \tag{3.8}$$

where

$$L_0(\beta) \approx \log_2 \chi(\beta) + C \sim A^{-1}(\beta_c - \beta)^{-1} \qquad \text{as } \beta \to \beta_c - \tag{3.9}$$

with $A \approx 2{\cdot}0 \pm 0{\cdot}2$ as before and $C \approx 1{\cdot}10$. At $\beta \approx \beta_c$, $C(p)$ decreases very slowly with increasing p, with data indicating, as shown in figure 3, that

$$C(p) \sim (\ln p)^{-1} \qquad \text{at } \beta = \beta_c \text{ as } p \to \infty. \tag{3.10}$$

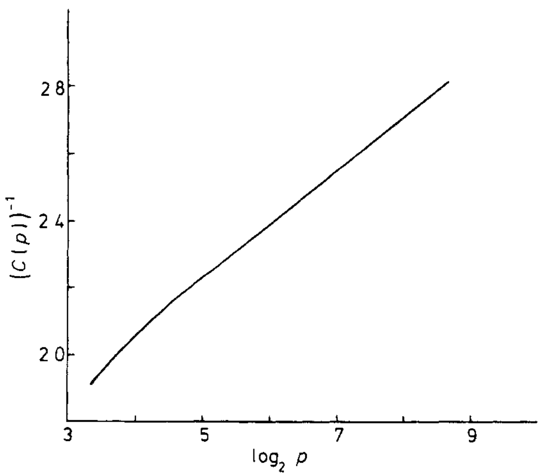

Figure 3. $C(p)^{-1}$ against $\log_2 p$ at $\beta = 1{\cdot}71155$.

For $\beta > \beta_c$, $C(p) \to m_0^2$, defined by (2.15), as $p \to \infty$. Typical behaviour of $C(p)$ is shown in figure 4. We note here that the low-temperature susceptibility is given by

$$\chi = 1 + \sum_{p=1}^{\infty} 2^{p-1}(C(p) - m_0^2) \tag{3.11}$$

so that for χ to exist for $\beta > \beta_c$, $C(p) - m_0^2$ must decrease faster than 2^{-p} as $p \to \infty$. Numerical data for $C(p)$, however, indicate that this is not the case and hence we conclude that χ is infinite for $\beta > \beta_c$.

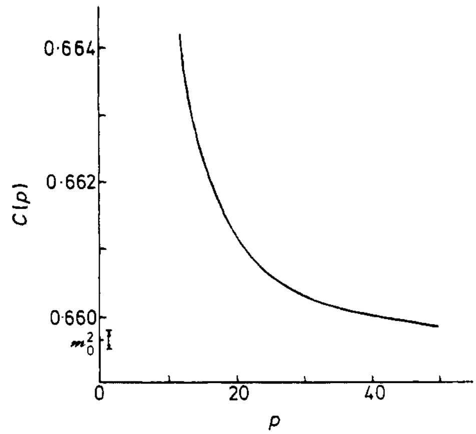

Figure 4. $C(p)$ against p at $\beta = 1{\cdot}80$.

4. Renormalisation group and essential singularities

When b_p in equation (1.2) is given by

$$b_p = 2^{(1-\sigma)p} \tag{4.1}$$

an exact renormalisation group transformation can be achieved by simply rescaling the spin distribution function or the partition function (Kim and Thompson 1977). Detailed critical behaviour of the model, as discussed in Kim and Thompson (1977), is essentially determined by the largest eigenvalue Λ_1 of the RGT linearised around the appropriate physical fixed point. In particular we found that as $\sigma \to 1-$, Λ_1 approached the marginal value of unity from above. Although there is, strictly speaking, no phase transition when σ is set equal to unity in (4.1) one can obtain a situation where there is a phase transition by considering

$$b_p = (1-\sigma)^{-1}(2^{(1-\sigma)p} - 1) \tag{4.2}$$

and taking the limit $\sigma \to 1-$, to obtain $b_p = p \ln 2$ and with Λ_1 still equal to its marginal value of unity. Similarly, by considering

$$b_p = 2^{(1-\sigma)p} \ln p, \tag{4.3}$$

which is more difficult to analyse than (4.1) or (4.2), we still expect that in the borderline limit $\sigma \to 1-$ the appropriate maximum eigenvalue becomes marginal. We claim that in general, when $\Lambda_1 = 1$, one can expect essential singularities.

To see this, consider a general renormalisation group equation

$$\pi_{l+1} = \mathbb{R}_l \circ \pi_l(y) \tag{4.4}$$

where $\mathbb{R}_l$ is a non-linear RGT that reduces the number of degrees of freedom by a factor of two and, in the present instance, $\pi_l(y)$ is either a spin distribution function or a scaled partition function. At criticality $\beta = \beta_c$, $\pi_l(y) \to \pi^*(y)$ as $l \to \infty$, where π^* satisfies

$$\pi^*(y) = \mathbb{R}_\infty \circ \pi^*(y). \tag{4.5}$$

Now let

$$\pi_l(y)=\pi^*(y)+\sum_{k=1}^{\infty}a_{k,l}(t)\psi_k(y); \qquad t=(\beta_c-\beta)/\beta_c \tag{4.6}$$

where $\psi_k(y)$ are the eigenfunctions of the linearised operator $\mathbb{L}$ (assumed to be compact) of $\mathbb{R}_\infty$,

$$\mathbb{L}\circ\psi_k(y)=\Lambda_k\psi_k(y); \qquad \Lambda_1=1>\Lambda_2>\Lambda_3>\dots. \tag{4.7}$$

For sufficiently large l and small t the 'irrelevant operators' ψ_k, $k>1$ in (4.6) are unimportant so that only the $k=1$ terms need to be considered. On substituting (4.6) into (4.4) and assuming that the non-linear contribution is quadratic (which is certainly the case in (2.5) for example) the RGT (4.4), in terms of the $a_{1,l}(t)$, becomes

$$a_{1,l+1}(t)\approx a_{1,l}(t)+A(a_{1,l}(t))^2+\text{higher order in } a_{1,l}(t). \tag{4.8}$$

Here A is some constant and from analyticity for finite l, $a_{1,l}(t)$ is small and proportional to t for small t. Iterating (4.8) n times we then have

$$a_{1,l+n}(t)\approx a_{1,l}(t)(1+nAa_{1,l}(t))\approx a_{1,l}(t)(1-nAa_{1,l}(t))^{-1} \tag{4.9}$$

so long as $nAa_{1,l}(t)\ll 1$. In particular if we set $a_{1,l}(t)=Bt$ we have

$$a_{1,l+n}(t)\approx Bt(1-nct)^{-1}; \qquad nct\ll 1. \tag{4.10}$$

For comparison, when $\Lambda_1>1$, which is the usual situation, one has (Wegner 1976)

$$a_{1,l+n}\approx a_{1,l}(t)\Lambda_1^n\approx Bt\Lambda_1^n \tag{4.11}$$

so that $a_{1,l}(t)$ deviates *exponentially* from its fixed point value zero, as l increases, whereas when $\Lambda_1=1$ it deviates only *linearly*. This linear behaviour is in fact observed numerically for the borderline HM, as shown in figure 5.

We can now proceed in the usual way to deduce singular behaviour from (4.10). Firstly, for the correlation length ξ, since each step of the RGT reduces the number of degrees of freedom by a factor of two, we have

$$\xi(\{a_{1,l}(t)\})=2^n\xi(\{a_{1,l+n}(t)\}),$$

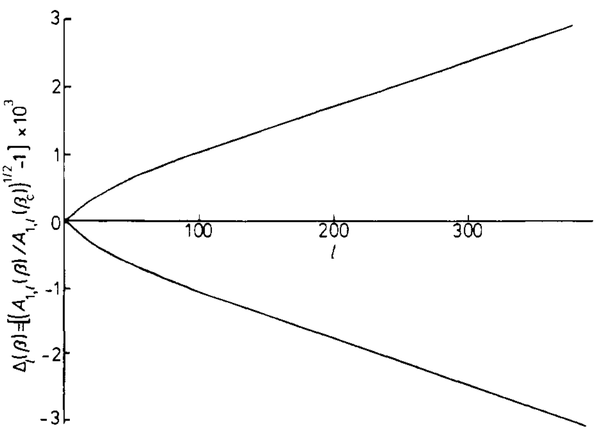

Figure 5. Behaviour of $A_{k,l}(\beta)$ defined in (2.6) relative to $A_{k,l}(\beta_c)$ near β_c for $k=1$. Upper data are for $\beta=1{\cdot}7116$ and lower data are for $\beta=1{\cdot}7115$ with β_c chosen as $\beta_c=1{\cdot}71155$.

that is, from (4.10)

$$\xi(t)=2^n\xi(t(1-nct)^{-1}); \qquad nct \ll 1. \tag{4.12}$$

Defining

$$f(x)=\ln \xi(x^{-1}) \tag{4.13}$$

we then have, from (4.12), that

$$f(x)=n \ln 2+f(x-nc) \tag{4.14}$$

for any n so long as $nc \ll x$, so that for sufficiently large x

$$f(x)\approx xc^{-1} \ln 2+D \tag{4.15}$$

and hence from (4.13)

$$\xi(t)\sim 2^{(ct)^{-1}} \qquad \text{as } t\to 0+. \tag{4.16}$$

For the HM, $\chi(t)\sim(\xi(t))^\sigma$ (Kim and Thompson 1977) so that for the borderline case one would expect the form (1.4) obtained numerically for χ. In the normal HM range $0<\sigma<1$ where (4.11) rather than (4.10) is valid, (4.12) is replaced by $(n=1)$

$$\xi(t)=2\xi(\Lambda_1 t) \tag{4.17}$$

so that assuming power law singularities $\xi\sim t^{-\nu}$ and $\chi\sim t^{-\gamma}$ one has

$$\nu=(\log_2 \Lambda_1)^{-1} \qquad \text{and} \qquad \gamma=\sigma(\log_2 \Lambda_1)^{-1}. \tag{4.18}$$

The corresponding expression to (4.17) for the spontaneous magnetisation m_0, when $0<\sigma<1$ is

$$m_0(-t)=2^{(\sigma-1)/2}m_0(-\Lambda_1 t) \tag{4.19}$$

so that if $m_0(t)\sim|t|^\beta$ as $t\to 0-$,

$$\beta=\tfrac{1}{2}(1-\sigma)(\log_2 \Lambda_1)^{-1}. \tag{4.20}$$

Assuming that (4.19) remains valid as $\sigma\to 1-$, with (4.10) replacing (4.11), one might expect that, granted the obvious difference between cases (4.1), (4.2) and (4.3), the borderline case $b_p=\ln p$ has spontaneous magnetisation $m_0(t)$ satisfying to first approximation, the functional equation

$$m_0(|t|)=m_0(|t|(1-nc|t|)^{-1}); \qquad nc|t|\ll 1. \tag{4.21}$$

Granted (4.21), and fixing ϵ sufficiently small, choose n (large) and $|t|$ (small) such that

$$nc|t|=\epsilon(1+\epsilon)^{-1}. \tag{4.22}$$

Equation (4.21) then becomes

$$m_0(|t|)=m_0((1+\epsilon)|t|). \tag{4.23}$$

Expanding the right-hand side of (4.23) in powers of $\epsilon|t|$ and allowing $\epsilon\to 0+$ one then obtains

$$tm_0'(t)=0 \qquad \text{as } t\to 0-. \tag{4.24}$$

It follows immediately that if $m_0(-c)>0$ with $0<c\ll 1$ then $m_0(0-)>0$ or in other words m_0 has a jump discontinuity.

The above discussion is of course incomplete and non-rigorous but at least it is consistent with our numerical results and Dyson's theorem (Dyson 1971) that the borderline HM has a first-order phase transition. In more general situations, where the usual RG result (4.11) breaks down due to the fact that $\Lambda_1 = 1$, it is most likely that essential singularities replace the conventional algebraic singularities though not necessarily of the form (4.16) which relies on the special form (4.8) of the non-linear RGT. The coincident first-order transition result above relies even more (equation (4.19)) on the HM structure of the RGT.

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